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# Tomographic representation of spin and quark states

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**Abstract**

We present a short review of the general principles of constructing tomograms of quantum states. We derive a general tomographic reconstruction formula for the quantum density operator of a system with a dynamical Lie group. In the reconstruction formula, the multiplicity of irreducible representation in Clebsch–Gordan decomposition is taken into account. Various approaches to spin tomography are discussed. An integral representation for the tomographic probability is found and a contraction of the spin tomogram to the photon-number tomography distribution is considered. The case of $SU(3)$ tomography is discussed with the examples of quark states (related to the simplest triplet representations) and octet states.

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## 1. Introduction

In order to reconstruct quasi-distributions such as the Wigner function, quantum tomography methods use conventional tomographic probability distributions (tomograms) [1, 2]. Methods have been proposed for the reconstruction of the density matrix (quasi-distributions) of the quantum states [3–6] including the states of spin-like systems [7–11]. The idea of quasi-distribution functions for angular momentum systems was introduced long ago [12, 13]. From an analysis of different types of tomogram, the idea of reformulation of quantum mechanics has appeared [14–16]. In this new formulation of quantum mechanics, which is closely connected with the tomography method, a quantum state is described by a probability distribution, alternative to the wavefunction or density matrix.

Tomography of quantum states in some cases can be related to a group representation or to other algebraic constructions [17, 18]. The tomography scheme proposed in [19, 20] was related to the symplectic group. The relation of spin tomography and symplectic tomography to the $SU(2)$ group and to the Heisenberg–Weyl group has been discussed in [21] and [22], respectively. The tomographic reconstruction scheme related to the $SU(1, 1)$ group has been studied in [23]. A detailed discussion of the relation of groups to tomography has been presented in [11, 24, 25]. In connection with signal analysis, a general construction of tomographic schemes and the relation of tomograms to quasi-distributions and wavelets has been proposed in [26].

In quantum mechanics, the star-product of functions is a method to map a product of operators on to a special product of functions. It is useful to formulate quantum evolution operators in terms of the evolution of functions, and the use of the Wigner function [27] within the framework of the Moyal equation [28] gives a realization of the star-product quantization procedure of classical systems. The spin tomograms have been considered within the framework of star-product quantization in [22]. A general scheme to map the dynamics, formulated in terms of operators, on to dynamics formulated in terms of functions of $c$-number arguments has been suggested by Stratonovich [12]. As shown in [29], the new formulation of quantum mechanics using the tomographic approach of [14–16] can be embedded into the standard star-product quantization procedure.

The aim of this paper is to study relations between different tomographic approaches to the spin-state description and to obtain the tomograms of quark states proposing the tomographic approach related to the $SU(3)$ group. Firstly, we discuss a tomographic symbol (tomogram) of an operator for the $SU(2)$ group [22]. The symbol is a function on the sphere. Also, we review a map of density operators acting in space of irreducible representations of the $SU(2)$ group on to set of probability distributions for projections of spin considered in different reference frames. One can point out that the relation of the spin tomogram to spin quasi-distributions of the $P$-distribution type [30] was discussed recently in [31]. The complete tomographic probability distribution contains redundant information on the spin density matrix. The non-redundant tomography scheme, which uses a finite number of data, has been presented by Weigert [32]. One can point out that the redundant information on the density operator can be useful in a real experimental situation. This information provides the possibility of controlling the consistency of experimental data.

On the other hand, the scheme of [10] provides the possibility of using a positive function on the sphere to reconstruct the spin density matrix as well. Here we show that this positive function is connected with the spin tomogram [7, 8]. It turns out that the positive function [10] equals one of the values of spin projection probabilities which determines all the other values. This means that the spin tomogram is similar to a distribution which contains only one parameter and the single probability value determines all the other values of the probabilities in the distribution function. There are many such one-parameter distributions, such as the thermal (Planck's) distribution, Poisson distribution and squeezed-vacuum distribution.

It is worth noting that, although a general structure of state-reconstruction methods for quantum systems possessing a group symmetry has been extensively discussed [11, 24, 25], the most attention has been paid to the Heisenberg–Weyl, symplectic and $SU(2)$ groups. On the other hand, there are physical states related to other groups, for example, to the $SU(3)$, $SU(4)$ and $SU(6)$ groups, etc. The dynamical groups describe the spectra of quantum systems [33]. The tomographic description of the rotational state of a top is given in [34].

In nuclear physics, the standard problem is measuring the spin density matrix of a nucleus using different types of experiment, such as, for example, $\gamma$–$\gamma$ correlations in cascade transitions or angular distributions of $\gamma$-quantum in the $(p, p\prime\gamma)$ process (with respect to the scattered proton) [35]. In [32] the reconstruction of the spin density matrix is suggested by using a finite number of measurements corresponding to several orientations of the system. In this case, it should be noted that in reactions of a $(p, p'\gamma)$ type the spin density matrix can be measured at different scattering angles of $p'$-proton, i.e. at different orientations of the reference frame axes because the $z$-axis is oriented along the transferred momentum. Thus, the used procedure is analogous to the discussed spin tomography procedure where the diagonal matrix elements of the density matrix in the rotated reference frame are necessary to reconstruct the complete density matrix. Both procedures, in which arbitrary angles [7, 10] or discrete angles [32] can be realized, are such types of nuclear experiment.

One of the aims of this paper is to extend a scheme of tomographic reconstruction of the Hermitian density matrix for the quantum states which are associated with irreducible representations of higher unitary groups (we focus on the case of simple compact groups). These representations are closely related to states of nuclei and elementary particles. Until now, the tomographic methods of measuring the quantum states have been developed in quantum optics only (see, for example, [36, 37]). Our aim is to point out that these methods can be applied also in nuclear and particle physics.

The paper is organized as follows. In section 2 we give a short review of the general principles of tomography schemes [38, 39]. In section 3 we derive a general tomographic reconstruction formula of the density matrix of a state related to a compact Lie group. We show that the reconstruction formula given in [11] should be corrected in the cases where multiplicities of the representations in decomposition of the tensor product of two representations are larger than unity (which is the most usual case for the $SU(3)$ group, for example). In section 4 we discuss in detail $SU(2)$ tomography. Due to a particular simplicity of this group, various reconstruction methods are possible.

We compare two different approaches (proposed in [7, 8], [10]) to spin tomography. Also, we find an integral representation for the tomographic probability and, in view of this, we analyse the contraction from spin to field systems (see [40, 41] for the contraction of $SU(2)$ quasi-distribution functions to the Heisenberg–Weyl case).

Extending the approach for higher groups, we consider in section 5 the case of $SU(3)$ tomography. This means that we describe the elementary particle states using probability distributions instead of vectors in Hilbert spaces. The superposition principle of quantum states was formulated in terms of tomographic probabilities (also in terms of density operators) in [42, 43]. Firstly, we discuss an example of quark states related to the simplest triplet representations. Then, we study a more complicated situation of octet states.

## 2. General approach to quantum tomography

In this section, we give a short review of the general principles used to construct a tomography scheme for measuring quantum states [38].

Let us consider a quantum state described by the density operator $\hat{\rho}$, which is a non-negative Hermitian operator, i.e.

$$\hat{\rho}^\dagger = \hat{\rho} \qquad \mathrm{Tr}\,\hat{\rho} = 1 \tag{1}$$

and

$$\langle v|\hat{\rho}|v\rangle = \rho_{v,v} \geqslant 0. \tag{2}$$

We label the vector basis $|v\rangle$ in the space of pure quantum states by the multi-dimensional index $v = (v_1, v_2, \ldots, v_N)$, where the number $N$ shows the number of degrees of freedom of the system under consideration. Among indices $v_k$, $k = 1, \ldots, N$, there are continuous indices, such as position (or momentum), and discrete indices, such as spin projections. In this sense, the wavefunction $\psi(v) = \langle v|\psi\rangle$ of a pure state $|\psi\rangle$ depends both on continuous and discrete observables. Using the Hermitian projection operator

$$\hat{\Pi}_v = |v\rangle\langle v| \tag{3}$$

formula (2) can be rewritten in the following form:

$$\rho_{v,v} = \mathrm{Tr}\{\hat{\Pi}_v \hat{\rho}\}. \tag{4}$$

The physical meaning of the projector $\hat{\Pi}_v$ is that it extracts the state $|v\rangle$ with given $v$ (for example, with given position and spin projection), which is an eigenstate of the commuting Hermitian operators $\hat{V} = (\hat{V}_1, \hat{V}_2, \ldots, \hat{V}_N)$

$$\hat{V}_k|v\rangle = v_k|v\rangle. \tag{5}$$

Let us consider the case where in the space of states there exists a family of unitary operators $\hat{U}(\sigma)$ depending on the parameters $\sigma = (\sigma_1, \ldots, \sigma_M)$, that can be sometimes identified with group-representation operators. In these cases, the parameters $\sigma$ describe the group element. Let us introduce a 'transformed density operator'

$$\hat{\rho}_\sigma = \hat{U}(\sigma)\hat{\rho}\hat{U}^{-1}(\sigma). \tag{6}$$

Its diagonal elements are still non-negative probabilities (called tomograms)

$$\langle z|\hat{\rho}_\sigma|z\rangle = \langle\langle z|\hat{\rho}|z\rangle\rangle \equiv w(z,\sigma) \tag{7}$$

where $|z\rangle$ is one of the possible vectors $|v\rangle$, while the symbol $|z\rangle\rangle$ denotes the transformed (auxiliary) vectors

$$|z\rangle\rangle = \hat{U}^{-1}(\sigma)|z\rangle \tag{8}$$

which in turn are eigenstates of the transformed operators

$$\hat{Z} = \hat{U}^{-1}(\sigma)\hat{V}\hat{U}(\sigma). \tag{9}$$

For a normalized density operator, we have the normalization condition for the tomogram

$$\int \mathrm{d}z\, w(z,\sigma) = 1. \tag{10}$$

In the case of discrete indices, the integral in equation (10) is replaced by a sum over discrete variables.

Formula (7) can be interpreted as the probability density for the measurement of the observable $\hat{V}$ in an ensemble of transformed reference frames labelled by the index $\sigma$, if the state $\hat{\rho}$ is given. Along with this interpretation, one can also consider the transformed projector

$$\hat{\Pi}_z(\sigma) = \hat{U}^{-1}(\sigma)\hat{\Pi}_z\hat{U}(\sigma) = |z\rangle\rangle\langle\langle z| \qquad \mathrm{Tr}\,\hat{\Pi}_z(\sigma) = 1. \tag{11}$$

The explicit expression for the probability $w(z,\sigma)$ takes the form

$$w(z,\sigma) = \mathrm{Tr}\{\hat{\rho}\hat{\Pi}_z(\sigma)\} = \mathrm{Tr}\{\hat{\rho}|z\rangle\rangle\langle\langle z|\}. \tag{12}$$

In fact, tomograms are tomographic symbols of density operators [29] and the transformed projectors determine a set of operators defining the symbols. The tomography schemes are based on the possibility of finding the inverse of equation (12). If it is possible to solve equation (12) by considering the probability $w(z,\sigma)$ as a known function and the density matrix as an unknown operator, the quantum state can be described by the positive probability instead of the density matrix. In such cases, the inverse of equation (12) takes the form

$$\hat{\rho} = \int \mathrm{d}z\,\mathrm{d}\sigma\, w(z,\sigma)\hat{K}(z,\sigma). \tag{13}$$

Thus, there exists a family of operators $\hat{K}(z,\sigma)$ depending on both the variables $z$ and the parameters $\sigma$ such that the density operator is reconstructed, if the probability $w(z,\sigma)$ is known. The operators $\hat{K}(z,\sigma)$ in equation (13) together with the transformed projectors (11) determine the kernel of the star-product of tomographic symbols used in the star-product quantization procedure [29]. It is worth remarking that transformations $\hat{U}(\sigma)$ can form other algebraic constructions, which can have no group structure [17, 18]. The only condition for the existence of a tomography scheme is the possibility of inverting equation (12). Where the states $|z\rangle$ form a basis of an irreducible representation of some Lie group and $\hat{U}(\sigma)$ are groups transformation operators, one can make some further construction. In this case, the density operator can be expanded in the basis of irreducible tensors using the Clebsch–Gordan series. Then, the inversion procedure can be carried out by using properties of the Clebsch–Gordan coefficients. Here, we apply this approach to the $SU(2)$ and $SU(3)$ groups.

## 3. Systems with dynamical group structure

In the cases of optical tomography [2], symplectic tomography [18, 20], and spin tomography [7, 8, 16], the sets of transformations $\hat{U}(\sigma)$ have the structure of corresponding Lie groups (i.e. rotation group $O(2)$, symplectic group $Sp(2,R)$, and the $SU(2)$ group). For systems with group structure, kernel operators $\hat{K}(z,\sigma)$ in the inverse formula (13) can be found explicitly in terms of group variables.

Let us consider a quantum system related to a group $G$. We focus on the case where $G$ is a simple compact group. We denote as

$$|A,[m],t\rangle \tag{14}$$

the orthonormal set of the basis state vectors of a unitary finite-dimensional irreducible representation $D^A$ of the group $G$. Here, $A$ is a signature of the representation which can be labelled also by eigenvalues of Casimir operators. The complex label $[m]$ specifies the weight of the basis vector and the label $t$ denotes all the other quantum numbers. In particular, for the $SU(2)$ group, the single half-integer index $-S \leqslant m \leqslant S$, uniquely defines a state of the $2S+1$-dimensional representation of the group (the representation is defined with a single number $A = S$ which is the spin). For the case of the $SU(3)$ group, $[m]$ contains two quantum numbers

$$[m] = [Y, M] \tag{15}$$

where $Y$ is the hypercharge and $M$, $-T \leqslant M \leqslant T$, is a projection of isospin $T$, which plays the role of the quantum number $t$ and distinguishes states with equal weights. The representation is characterized by two numbers $p, q = 0, 1, 2, \ldots$, i.e.

$$A = (p, q).$$

If the ket-vector $|A,[m],t\rangle$ transforms according to the representation $A$ of the group $G$

$$D(g)|A,[m],t\rangle = \sum_{[m_1],t_1} D^A_{[m_1]t_1,[m]t}(g)|A,[m_1],t_1\rangle$$

the bra-vector $\langle A,[m],t|$ transforms according to the conjugate representation $A^*$ and has weight $[-m]$, i.e. the vector $\langle A,[m],t|$ transforms like the vector $(-1)^{y(A[m],t)}|A^*,[-m],t^*\rangle$, where the phase factor $(-1)^{y(A[m],t)}$ is a function of the group representation. For the $SU(2)$ group,

$$A^* = S \qquad \text{and} \qquad y(A[m],t) = S - m$$

while for the $SU(3)$ group

$$A^* = (q, p) \qquad \text{and} \qquad y(A[m], t) = -\frac{2p+q}{3} - Y + T - M$$

and isospin $T$ does not change its value under conjugation.

The system density matrix can be expanded in the basis (14) as follows:

$$\rho = \sum_{[m],t} \sum_{[m_1],t_1} \rho_{[m]t,[m_1]t_1} |A, [m], t\rangle\langle A, [m_1], t_1|. \tag{16}$$

The operator $|A, [m], t\rangle\langle A, [m_1], t_1|$ (projector) transforms according to the direct product of representations $A \otimes A^*$, which can be expanded in the direct sum of irreducible representations according to the symbolic formula

$$A \otimes A^* = \sum_B \oplus k_B B \tag{17}$$

where $k_B$ is the (outer) multiplicity of the representation $B$ in the above tensor product. Note that, in the case of the $SU(2)$ group, $k_B$ is always unity, which is not true, for example, for the $SU(3)$ group. Thus, the density matrix (16) can be represented in the form of expansion in terms of irreducible tensors $T^{B,s}_{[m_1]t_1}$, which form a basis in the space of operators acting in the irreducible representation $B$

$$\rho = \sum_{B,[m_1],t_1} \sum_{s=1}^{k_B} R^{B,s}_{[m_1]t_1} T^{B,s}_{[m_1]t_1} \tag{18}$$

where the index $B$ runs over all the values appearing in the expansion (17). The operators $T^B_{[m_1]t_1}$ transform like elements of the basis $|B, [m], t\rangle$

$$D(g) T^B_{[m_1]t_1} D^\dagger(g) = \sum_{[m_2],t_2} D^B_{[m_2]t_2,[m_1]t_1}(g) T^B_{[m_2]t_2}$$

and have the following matrix elements:

$$\langle A[m]t| T^{B,s}_{[m_1]t_1} |A[m_2]t_2\rangle = (-1)^{y(A[m_2],t_2)} \langle A[m]t;\, A^*[-m_2]t_2^* | B[m_1]t_1\rangle_s \tag{19}$$

where $\langle A[m]t;\, B[m_1]t_1 | C[m_2]t_2\rangle_s$ are the Clebsch–Gordan coefficients which couple two irreducible representations $A$ and $B$ to the representation $C$. The operators $T^{B,s}_{[m_1]t_1}$ satisfy the following normalization condition,

$$\mathrm{Tr}\left[ T^{A,s}_{[m]t} \left( T^{B,s_1}_{[m_1]t_1} \right)^\dagger \right] = \delta_{AB} \delta_{[m][m_1]} \delta_{ss_1} \delta_{tt_1}$$

and the reduced matrix elements of $T^{B,s}_{[m_1]t_1}$ are chosen to be unity.

Thus, we can express coefficients $R^{B,s}_{[m_1]t_1}$ in terms of elements of the density matrix $\rho_{[m]t,[m_1]t_1}$ as follows:

$$\begin{aligned} R^{B,s}_{[m_1]t_1} &= \mathrm{Tr}\left( \rho \left( T^{B,s_1}_{[m_1]t_1} \right)^\dagger \right) \\ &= \sum_{[m],t} \sum_{[m_2],t_2} (-1)^{y(A[m_2],t_2)} \langle A[m]t;\, A^*[-m_2]t_2^* | B[m_1]t_1\rangle_s \rho_{[m]t,[m_2]t_2}. \end{aligned}$$

The inverse relation reads

$$\rho_{[m]t,[m_2]t_2} = \sum_{B,[m_1],t_1,s} (-1)^{y(A[m_2],t_2)} \langle A[m]t;\, A^*[-m_2]t_2^* | B[m_1]t_1\rangle_s R^{B,s}_{[m_1]t_1}.$$

According to the general scheme, the transformed density operator (6) has the form

$$\hat{\rho}_g = D(g)\hat{\rho}D^{\dagger}(g) = \sum_{B,[m_1],t_1,s}\sum_{[m_2],t_2} R^{B,s}_{[m_1]t_1} D^{B}_{[m_2]t_2,[m_1]t_1}(g) T^{B,s}_{[m_2]t_2} = \sum_{B,[m_2],t_2,s} \tilde{R}^{B,s}_{[m_2]t_2} T^{B,s}_{[m_2]t_2}$$

where

$$\tilde{R}^{B,s}_{[m_2]t_2} = \sum_{[m_1],t_1} R^{B,s}_{[m_1]t_1} D^{B}_{[m_2]t_2,[m_1]t_1}(g).$$

The probability of detecting the transformed system in a state $|A,[m],t\rangle$ for a fixed group parameter $g$ is given by equation (12), i.e.

$$w(g, A[m]t) = \langle A[m]t|\hat{\rho}_g|A,[m],t\rangle \sum_{B,s,t_2}\sum_{[m_1],t_1} R^{B,s}_{[m_1]t_1} D^{B}_{[0]t_2,[m_1]t_1}(g) \times (-1)^{y(A[m],t)}\langle A[m]t;\, A^*[-m]t^*|B[0]t_2\rangle_s \tag{20}$$

where we have used equation (19). It is clear that the state tomogram $w(g, A[m]t)$ is normalized according to

$$\int \mathrm{d}g w(g, A[m]t) = \frac{V_G}{\dim A} \qquad \sum_{[m],t} w(g, A[m]t) = 1. \tag{21}$$

In equation (21) the group Haar measure is used. Note that equation (20) can be rewritten in the form (12)

$$w(g, A[m]t) = \mathrm{Tr}(\rho\hat{\Pi}_{[m]t}(g))$$

where the transformed projector $\hat{\Pi}_{[m]t}(g)$ (11) is defined as

$$\hat{\Pi}_{[m]t}(g) = \sum_{B,[m_1],t_1,s}\sum_{[m_2],t_2} \left(T^{B,s}_{[m_1]t_1}\right)^{\dagger} D^{B}_{[0]t_2,[m_1]t_1}(g)(-1)^{y(A[m],t)} \times\langle A[m]t;\, A^*[-m]t^*|B[0]t_2\rangle_s. \tag{22}$$

Multiplying both sides of equation (20) by the matrix element $\left(D^{B}_{[0]t_2',[m_1']t_1'}(g)\right)^*$, integrating over the group and making use of the orthogonality relation for matrix elements of irreducible representations of the compact group

$$\int \mathrm{d}g D^{B}_{[m]t,[m_1]t_1}(g)\left(D^{C}_{[m']t',[m_1']t_1'}(g)\right)^* = \frac{V_G}{\dim B}\delta_{CB}\delta_{[m_1'][m_1]}\delta_{[m'][m]}\delta_{tt'}\delta_{t_1t_1'}$$

where $\mathrm{d}g$ is an invariant measure on the group and $V_G = \int \mathrm{d}g$ is the group volume, we obtain

$$\sum_s (-1)^{y(A[m],t)} R^{B,s}_{[m_1]t_1}\langle A[m]t;\, A^*[-m]t^*|B[0]t_2\rangle_s = \frac{\dim B}{V_G}\int \mathrm{d}g \left(D^{B}_{[0]t_2,[m_1]t_1}(g)\right)^* w(g, A[m]t). \tag{23}$$

In the absence of multiplicity, $k_B = 1$, we obtain from equation (23) coefficients of expansion of the system density matrix in terms of the irreducible tensor operators (18)

$$R^{B,s}_{[m_1]t_1} = \frac{(-1)^{y(A[m],t)}\dim B}{V_G\langle A[m]t;\, A^*[-m]t^*|B[0]t_2\rangle_s}\int \mathrm{d}g \left(D^{B}_{[0]t_2,[m_1]t_1}(g)\right)^* w(g, A[m]t). \tag{24}$$

A similar expression has been previously found in [11] and, in the particular case of the $SU(2)$ group, in [10]. Nevertheless, in the case where $k_B \geqslant 2$, the above expression, in general, is not correct and to find coefficients $R^{B,s}_{[m_1]t_1}$, $s = 1, \ldots, k_B$ corresponding to degenerate representations (which enter more than once in the expansion (17)) we have to solve the system of linear equations (23) for all possible values of index $t_2$. Representing coefficients $R^{B,s}_{[m_1]t_1}$ as a $k_B$ dimensional vector $\vec{R}$, we find from equation (23)

$$\vec{R} = C^{-1}\vec{J} \tag{25}$$

where

$$(\vec{J})_{t_2} = \frac{\dim B}{V_G}\int \mathrm{d}g\left(D^B_{[0]t_2,[m_1]t_1}(g)\right)^* w(g, A[m]t)$$

and the $k_B \times k_B$-matrix $C$ has the following matrix elements:

$$(C)_{t_2 s} = (-1)^{y(A[m],t)}\langle A[m]t;\, A^*[-m]t^*|B[0]t_2\rangle_s. \tag{26}$$

Let us recall that the index $t_2$ in the matrix $C$ enumerates states with the same values of the composite index $[m]$, i.e. $t_2$ is the index of internal multiplicity. Because the outer multiplicity $k_B$ is less than (or equal to) the inner multiplicity of the state corresponding to the weight $[m = 0]$, we can always choose such a set of indices $t_2$, that the matrix $C$ will be non-degenerate. Usually in equations (23) and (24) the highest weight vector of the irreducible representation $A$ is used as the reference vector $|A, [m], t\rangle$. We discuss the problem of outer multiplicity using the example of the $SU(3)$ group.

## 4. Two reconstruction formulae for the $SU(2)$ group

In this section, we apply the results of the previous sections to the $SU(2)$ group (spin-$S$ systems) and obtain different forms of inverse formulae for the density matrix in terms of spin tomograms $w(z, \sigma)$ (7). In this section, we consider the $2S+1$-dimensional representation of the $SU(2)$ group. The basis of the representation space is formed by eigenstates $|k, S\rangle$ of the spin-projection operator $S_z$

$$S_z|k, S\rangle = k|k, S\rangle \qquad k = -S, \ldots, S.$$

In this case, $\dim A = 2S + 1$ and $V_G = 8\pi^2$.

### *4.1. First reconstruction formula*

The tomogram for the spin-$S$ system is defined as follows,

$$w_k = \langle k, S|D^{(S)}(g)\rho D^{(S)\dagger}(g)|k, S\rangle \tag{27}$$

where $\rho$ is the density matrix and

$$D^{(S)}(g) = D^{(S)}(\psi, -\theta, -\phi) \tag{28}$$

(for convenience, the signs of the Euler angles $\theta$ and $\phi$ are chosen to be negative) is the finite rotation operator in the Euler parametrization, such that

$$D^{(L)}_{0M}(\psi, -\theta, -\phi) = \sqrt{\frac{4\pi}{2L+1}}Y_{LM}(\theta, \phi).$$

It should be remembered that the physical meaning of the tomogram is that it is equal to the probability of obtaining the spin projection $k$ of the total spin $S$ on the axis determined by the Euler angles $\theta$ and $\varphi$. The transformed projector (11) takes the form (where the signs of the Euler angles $\theta$ and $\phi$ are taken according to the parametrization (28))

$$\hat{\Pi}_k(g) = \hat{\Pi}_k(\theta, \phi) = \sqrt{\frac{4\pi}{2S+1}}\sum_{L=0}^{2S}\sum_{M=-L}^{L}\hat{T}^{(S)\dagger}_{L,M}\langle Sk;\, L0|Sk\rangle Y_{LM}(\theta, \phi) \tag{29}$$

where $\hat{T}^{(S)}_{L,M}$ is the irreducible tensor operator for the $SU(2)$ group

$$\hat{T}^{(S)}_{L,M} = \sqrt{\frac{2L+1}{2S+1}} \sum_{m,m'=-S}^{S} \langle Sm; LM|Sm'\rangle |S, m'\rangle\langle S, m| \tag{30}$$

and

$$\langle Sm; LM|Sm'\rangle = (-1)^{S-m}\sqrt{\frac{2S+1}{2L+1}}\langle Sm'; S-m|LM\rangle$$

is the Clebsch–Gordan coefficient which connects two representations of spin $S$ with the representation determined by total spin $0 \leqslant L \leqslant 2S$. The quantum number $M$ is the total spin projection. It is easy to see that the tomographic probability (27)

$$w_k = w_k(\theta, \phi) = \mathrm{Tr}(\rho \hat{\Pi}_k(\theta, \phi))$$

takes the form

$$w_k = \sqrt{\frac{4\pi}{2S+1}} \sum_{L=0}^{2S} \sum_{M=-L}^{L} \mathrm{Tr}\left(\hat{T}^{(S)\dagger}_{L,M}\rho\right) \langle Sk; L0|Sk\rangle Y_{LM}(\theta, \phi). \tag{31}$$

This satisfies the following normalization conditions,

$$\frac{2S+1}{4\pi}\int \mathrm{d}\Omega w_k(\theta, \phi) = 1 \qquad \sum_{k=-S}^{S} w_k(\theta, \phi) = 1 \tag{32}$$

where $\mathrm{d}\Omega = \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi$.

For example, for the density matrix of an arbitrary pure spin-1/2 state (which is determined by a point on the two-dimensional sphere)

$$\rho_{\vec{n}} = \begin{bmatrix} \cos^2\vartheta & \mathrm{e}^{-\mathrm{i}\varphi}\cos\vartheta\sin\vartheta \\ \mathrm{e}^{\mathrm{i}\varphi}\cos\vartheta\sin\vartheta & \sin^2\vartheta \end{bmatrix} \tag{33}$$

the tomographic probability (27) takes the form

$$w_{1/2}(\theta, \phi) = \cos^2\frac{\theta}{2}\cos^2\vartheta + \sin^2\frac{\theta}{2}\sin^2\vartheta + \frac{1}{2}\sin\theta\cos(\phi - \varphi)\sin 2\vartheta$$
$$w_{-1/2}(\theta, \phi) = 1 - w_{1/2}(\theta, \phi).$$

The density matrix $\rho$ can be expanded in terms of the irreducible tensors (30) as follows:

$$\rho = \sum_{L=0}^{2S}\sum_{M=-L}^{L} \hat{T}^{(S)}_{L,M}\,\mathrm{Tr}\left(\hat{T}^{(S)\dagger}_{L,M}\rho\right) = \sum_{L=0}^{2S}\sum_{M=-L}^{L} \hat{T}^{(S)}_{L,M} R^{(S)}_{L,M}. \tag{34}$$

Thus, for the coefficients $R^{(S)}_{L,M}$, we obtain from equation (24)

$$R^{(S)}_{L,M} = \sqrt{\frac{2S+1}{4\pi}}\langle Sk; L0|Sk\rangle^{-1}\int \mathrm{d}\Omega Y^*_{LM}(\phi, \theta) w_k(\theta, \phi). \tag{35}$$

From the above equation it follows that the system density matrix can be reconstructed, in view of the tomographic probability distribution $w_k(\theta, \phi)$ for an arbitrary value of the spin projection $k$ [11]. In the particular case $k = -S$, the tomographic probability is the $Q$-function for spin systems [10]. It is worth noting that for $k = 0$ the Clebsch–Gordan coefficient $\langle Sk; L0|Sk\rangle$ equals zero when $2S + L$ is an odd number, and thus the inversion formula (35) does not work. This means that the reconstruction of the density matrix from $w_k$ is subtle and not all components in the expansion (34) can be reconstructed at $k = 0$, but only those for which $2S + L$ is an even number.

*4.2. Integral representation*

An integral representation for the operator $\hat{\Pi}_k(\theta, \phi)$ can be found in the same way as for the Stratonovich–Wigner operator [40] by making use of the following expansion for elements of the irreducible representation of the $SU(2)$ group [44],

$$\mathrm{e}^{-\mathrm{i}\vec{y}\cdot\vec{S}} = \frac{\sqrt{4\pi}}{\sqrt{2S+1}} \sum_{L=0}^{2S} \sum_{M=-L}^{L} (-\mathrm{i})^L \chi_L^S(\omega) Y_{L,M}^*(\vec{n}) \widehat{T}_{L,M}^{(S)} \tag{36}$$

where

$$\vec{y} = \omega\vec{n} \qquad \vec{n} = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta) \qquad \vec{S} = (S_x, S_y, S_z)$$

are the $su(2)$ algebra operators and $\chi_L^S(\omega)$ are the generalized characters of the group [44]

$$\chi_L^S(\omega) = \mathrm{i}^L \sum_M \mathrm{e}^{-\mathrm{i}M\omega} \langle SM; L0|SM\rangle \tag{37}$$

which obey the following orthogonality relation (following from the above formula and sum rules for the Clebsch–Gordan coefficients):

$$\int_0^{2\pi} \mathrm{d}\omega \chi_L^S(\omega) \chi_{L'}^S(\omega) = 2\pi \delta_{L,L'} \frac{2S+1}{2L+1}. \tag{38}$$

Using the orthogonality relation (38) one can invert the expansion (36) giving the following integral representation for the operator (29) (similar to that for the Stratonovich–Wigner operator [40]),

$$\hat{\Pi}_k(\phi, \theta) = \int_0^{2\pi} \mathrm{d}\omega\, \mathrm{e}^{\mathrm{i}\omega\vec{n}\cdot\vec{S}} f(\omega) \tag{39}$$

where the weight function $f(\omega)$ is defined as follows:

$$f(\omega) = \frac{1}{2\pi} \sum_{L=0}^{2S} (-\mathrm{i})^L \frac{2L+1}{2S+1} \chi_L^S(\omega) \langle Sk; L0|Sk\rangle = \frac{1}{2\pi}\, \mathrm{e}^{-\mathrm{i}\omega k}. \tag{40}$$

Thus, equation (39) takes a simple form

$$\hat{\Pi}_k(\theta, \phi) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\omega\, \mathrm{e}^{\mathrm{i}\omega(\vec{n}\cdot\vec{S}-k)} \tag{41}$$

and the tomographic probability function acquires the form analogous to the probability distribution introduced in [45] as the Fourier transform of the characteristic function for the observable $\vec{S}\vec{n}$,

$$w_k(\theta, \phi) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\omega\, \mathrm{Tr}(\mathrm{e}^{\mathrm{i}\omega\vec{n}\cdot\vec{S}} \rho)\, \mathrm{e}^{-\mathrm{i}\omega k}. \tag{42}$$

Using the expansion (36) one can rewrite the reconstruction formulae (34) and (35) in the integral form dual to (42)

$$\rho = \int \mathrm{d}\Omega \int_0^{2\pi} \mathrm{d}\omega\, \mathrm{e}^{-\mathrm{i}\omega\vec{n}\cdot\vec{S}} g(\omega) w_k(\theta, \phi) \tag{43}$$

where

$$g(\omega) = \frac{1}{8\pi^2} \sum_{L=0}^{2S} \mathrm{i}^L (2L+1) \chi_L^S(\omega) \langle Sk; L0|Sk\rangle^{-1}.$$

*4.3. Spin tomogram and density operator*

The general expression (24) and its particular consequences (34), (35) and (43) represent reconstruction formulae, where the density matrix is determined using the single tomographic probability value as a function on the sphere (12). In the case of the $SU(2)$ group, another reconstruction scheme is possible. Sometime, due to errors while measuring the tomogram $w(z, \sigma)$, more than one value of probability of the observable is desirable for additional control. In this section, we derive such expressions for the density matrix reconstruction.

Let us note that from equation (31) and the expansion (36) together with the definition of generalized characters (37) it follows that

$$\sum_{k=-S}^{S} w_k(\theta, \phi)\,\mathrm{e}^{\mathrm{i}\omega k} = \mathrm{Tr}(\mathrm{e}^{\mathrm{i}\omega\vec{n}\cdot\vec{S}}\rho).$$

Making use of the expansion

$$\hat{L} = \frac{2S+1}{4\pi^2}\int \mathrm{d}V\ \mathrm{Tr}(\mathrm{e}^{\mathrm{i}\omega\vec{n}\cdot\vec{S}}\hat{L})\,\mathrm{e}^{-\mathrm{i}\omega\vec{n}\cdot\vec{S}}$$

where $\mathrm{d}V = \sin^2\omega/2\,\mathrm{d}\omega\,\mathrm{d}\Omega$ and $\hat{L}$ is an arbitrary $(2S+1)\times(2S+1)$ matrix, we obtain the following inversion formula (see also [46]):

$$\rho = \frac{2S+1}{4\pi^2}\int \mathrm{d}V\,\mathrm{e}^{-\mathrm{i}\omega\vec{n}\cdot\vec{S}}\sum_{k=-S}^{S} w_k(\theta, \phi)\,\mathrm{e}^{\mathrm{i}\omega k}. \tag{44}$$

Using the normalization condition (32), one can prove that the condition $\mathrm{Tr}\rho = 1$ is preserved. Note that in the above reconstruction formula all the probabilities $w_k(\theta, \phi)$ are involved. Equation (44) has a clear physical sense and can be interpreted as a double (operational and finite) Fourier transformation of the tomographic probabilities $w_k(\theta, \phi)$.

One can also find another reconstruction formula for the density matrix. Multiplying equation (31) by $\langle Sk;\, L0|Sk\rangle Y^*_{LM}(\theta, \phi)$, integrating over the solid angle $\Omega$ and summing over the index $k$, we obtain

$$\mathrm{Tr}\left(\hat{T}^{(S)\dagger}_{L,M}\rho\right) = \frac{2L+1}{\sqrt{4\pi(2S+1)}}\int \mathrm{d}\Omega Y^*_{LM}(\theta, \phi)\sum_{k=-S}^{S} w_k(\theta, \phi)\langle Sk;\, L0|Sk\rangle \tag{45}$$

where we have used the following sum rules [44]:

$$\sum_{k=-S}^{S}\langle Sk;\, L0|Sk\rangle\langle Sk;\, L_1 0|Sk\rangle = \frac{2S+1}{2L+1}\delta_{LL_1}.$$

Equations (34) and (45) immediately give the reconstruction formula

$$\rho = \sum_{L=0}^{2S}\sum_{M=-L}^{L}\hat{T}^{(S)}_{L,M}\frac{2L+1}{\sqrt{4\pi(2S+1)}}\int \mathrm{d}\Omega Y^*_{LM}(\theta, \phi)\sum_{k=-S}^{S} w_k(\theta, \phi)\langle Sk;\, L0|Sk\rangle$$

which has been obtained in [7, 8, 22] in a different way.

*4.4. Contraction to photon-number tomogram*

Taking into account the decomposition

$$\mathrm{e}^{-\mathrm{i}\omega\vec{n}\cdot\vec{S}} = D(\phi, \theta)\,\mathrm{e}^{-\mathrm{i}\omega S_z}D^\dagger(\phi, \theta) \tag{46}$$

where $D(\phi, \theta)$ is the displacement operator

$$D(\phi, \theta) = \exp\left[\frac{\theta}{2}(S_- e^{i\phi} - S_+ e^{-i\phi})\right] \tag{47}$$

one can obtain from equation (41) another representation for the operator $\hat{\Pi}_k(\theta, \phi)$,

$$\hat{\Pi}_k(\theta, \phi) = D(\phi, \theta)\hat{\sigma}_k D^\dagger(\phi, \theta) \tag{48}$$

where

$$\hat{\sigma}_k = \frac{1}{2\pi}\int_0^{2\pi} d\omega\, e^{-i\omega(S_z - k)} = \delta(S_z - k). \tag{49}$$

Expression (48) is very suitable for considering the limit $S \to \infty$ and obtaining contraction to the Heisenberg–Weyl group (see [40, 41]). Let us assume, that the system density matrix has a sharp maximum in the vicinity of the north pole of the sphere. In the basis of eigenvalues of $S_z$, this means that the only density matrix elements, essentially different from zero, are $\rho_{S-k,S-j}$, with $k, j \ll S$. On the other hand, in the displacement operator $D(\theta, \phi)$ one should take the limit $\theta \to 0$. Then we may substitute

$$S_+ = \sqrt{2S}a^\dagger \qquad S_- = \sqrt{2S}a \qquad S_z = \hat{N} - S \tag{50}$$

such that

$$[\hat{N}, a^\dagger] = a^\dagger \qquad [\hat{N}, a] = -a \qquad [a, a^\dagger] = 1 - \frac{\hat{N}}{S}$$

and the displacement operator (47) takes the form

$$D(\phi, \theta) = \exp\left[\theta\sqrt{\frac{S}{2}}(a^\dagger e^{i\phi} - a\, e^{-i\phi})\right].$$

In the limit where $\theta \to 0$ and $S \to \infty$, so that $r = \theta\sqrt{S/2}$ remains finite, we obtain

$$D(\phi, \theta) \to D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) \qquad \alpha = -r\, e^{i\phi} \tag{51}$$

which is just the displacement operator for the Heisenberg–Weyl group. To take the limit in the $\hat{\sigma}$ operator (49), we change the index $k$ (which originally indicates a state of the $su(2)$ spectrum counting from the bottom, $k = -S, \ldots, S$) to $n - S$, where $n = 0, \ldots, 2S$ indicates the position of the same state but counting from the top of the spectrum (from the north pole of the sphere). Substituting equation (50) into (49) and taking into account that $\hat{N} \to a^\dagger a$, we obtain

$$\hat{\sigma}_k \to \hat{\sigma} = \delta(\hat{N} - n)$$

and the operator $\hat{\Pi}_k(\phi, \theta)$ converts into

$$\hat{\Pi}_k(\phi, \theta) \to \hat{\Pi}_n(\alpha) = D(\alpha)\delta(\hat{N} - n)D^\dagger(\alpha)$$

where one can easily recognize the operator which generates the photon-number tomogram [3, 4, 17, 26]. In fact, we obtain

$$w_n(\alpha) = \mathrm{Tr}(D(\alpha)\delta(\hat{N} - n)D^\dagger(\alpha)\rho) = \langle n|D^\dagger(\alpha)\rho D(\alpha)|n\rangle$$

where $|n\rangle$ is the $n$-photon-number state. This probability distribution function [45] is just the photon-number tomogram of [3, 4, 17].

## 5. Transition probabilities

Using equations (34)–(35) one can find the transition probability (fidelity) between two spin states characterized by the density matrices $\rho_1$ and $\rho_2$,

$$\begin{aligned} W_{12} &= \mathrm{Tr}(\rho_1\rho_2) \\ W_{12} &= \int \mathrm{d}\Omega\,\mathrm{d}\Omega' w_k^{(1)}(\Omega) w_k^{(2)}(\Omega')\alpha_{k,S}(\Omega,\Omega') \end{aligned} \tag{52}$$

where $\Omega$ denotes the solid angle and

$$\alpha_{k,S}(\Omega,\Omega') = \frac{2S+1}{4\pi}\sum_{L,L'=0}^{2S}\sum_{M,M'=-L}^{L}(\langle Sk;\,L0|Sk\rangle\langle Sk;\,L'0|Sk\rangle)^{-1} \times Y^*_{LM}(\Omega)Y_{LM}(\Omega')\,\mathrm{Tr}\left(\hat{T}^{(S)}_{L,M}\hat{T}^{(S)\dagger}_{L',M'}\right).$$

Taking into account

$$\mathrm{Tr}\left(\hat{T}^{(S)}_{L,M}\hat{T}^{(S)\dagger}_{L',M'}\right) = \delta_{LL'}\delta_{MM'}$$

and using the summation rule

$$\sum_{M=-L}^{L} Y^*_{LM}(\Omega)Y_{LM}(\Omega') = \frac{2L+1}{4\pi}P_L(\cos\omega)$$

where

$$\cos\omega = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

and $P_L(\cos\omega)$ are the Legendre polynomials, we finally obtain

$$\alpha_{k,S}(\Omega,\Omega') = \frac{2S+1}{(4\pi)^2}\sum_{L=0}^{2S}\langle Sk;\,L0|Sk\rangle^{-2}(2L+1)P_L(\cos\omega). \tag{53}$$

Since the reconstruction formula used here has the same form for arbitrary spin projections $k$, the fidelity (52) has the same form for different $k$ as well. In particular, the purity parameter in terms of the tomographic probabilities takes the form

$$\mu = \mathrm{Tr}(\rho^2) = \int \mathrm{d}\Omega\,\mathrm{d}\Omega' w_k(\Omega)w_k(\Omega')\alpha_{k,S}(\Omega,\Omega').$$

In the case of $S = 1/2$, we obtain from equation (53)

$$\alpha_{k,S}(\Omega,\Omega') = \frac{2}{(4\pi)^2}(1+9\cos\omega) \tag{54}$$

which is independent of the index $k = \pm 1/2$. Substituting explicit expressions for the tomographic probabilities in terms of the Stokes parameters

$$2\rho_{1/2,-1/2} = \xi_1 + \mathrm{i}\xi_2 \qquad \rho_{1/2,1/2} - \rho_{-1/2,-1/2} = \xi_3$$

gives

$$w_{\pm 1/2}(\Omega) = \frac{1+\xi_3\cos\theta}{2} \pm \frac{1}{2}\,\mathrm{Re}[(\xi_1+\mathrm{i}\xi_2)\,\mathrm{e}^{\mathrm{i}\phi}]\sin\theta$$

and inserting equation (54) into (52) we obtain, after performing a trivial integration, the well-known result

$$W_{12} = \tfrac{1}{2}\left(1 + \xi_1^{(1)}\xi_1^{(2)} + \xi_2^{(1)}\xi_2^{(2)} + \xi_3^{(1)}\xi_3^{(2)}\right).$$

## 6. Tomography for the $SU(3)$ group

In this section, we apply general equations (23) and (24) to the case of the $SU(3)$ group. All necessary formulae of the theory of the $SU(3)$ group representations are given in appendices A–D.

### *6.1. Quark states*

There are two (conjugate) fundamental representations of the $SU(3)$ group, $(1, 0)$ and $(0, 1)$, which correspond to quark and anti-quark states. The dimension of these representations is $\dim A = 3$. The three basis vectors of the $(1, 0)$ representation (quark states) are of the form (15)

$$|1\rangle = |(1,0), [-2/3, 0], 0\rangle$$
$$|2\rangle = |(1,0), [1/3, 1/2], 1/2\rangle$$
$$|3\rangle = |(1,0), [1/3, -1/2], 1/2\rangle.$$

The basis of the conjugate representation (01) (anti-quark states) is formed by vectors

$$|\bar{1}\rangle = |(0,1), [2/3, 0], 0\rangle$$
$$|\bar{2}\rangle = |(0,1), [-1/3, -1/2], 1/2\rangle$$
$$|\bar{3}\rangle = |(0,1), [-1/3, 1/2], 1/2\rangle.$$

Their tensor product (17) can be decomposed into the direct sum of two irreducible representations

$$(1,0) \otimes (0,1) = (1,1) \oplus (0,0) \tag{55}$$

where $(1, 1)$ corresponds to the octet state and $(0, 0)$ is the singlet state. Because the expansion (55) is not degenerate (both states in this expansion enter with multiplicity $k_B = 1$) we can apply equation (24) directly in order to obtain coefficients $R^B_{[m_1]t_1}$ in expansion (18). For a reference state $|A[m]t\rangle$, we obtain

$$R^B_{[m_1]t_1} = \frac{(3\dim B)^{1/2}}{V_G\langle(10)[m]t;\, B[0]t_2|(10)[m]t\rangle} \int \mathrm{d}g\, \left(D^B_{[0]t_2,[m_1]t_1}(g)\right)^* w(g, (10)[m]t)$$

where

$$w(g; (10)[m]t) = |\langle(10)[m]t|D(g)|(10)[m]t\rangle|^2.$$

We take into account the following symmetry relation for the Clebsch–Gordan coefficients (which holds for this non-degenerate case and self-conjugate representation $B$):

$$(-1)^{y(A[m_2],t_2)}\langle A[m]t;\, A^*[-m_2]t_2^*|B[m_1]t_1\rangle = \sqrt{\frac{\dim B}{\dim A}}\langle A[m]t;\, B[-m_1]t_1|A[m_2]t_2\rangle.$$

The group volume is $V_G = (1/2)(4\pi)^5$. The index $B$ labelling representations takes values $(0, 0)$ and $(1, 1)$ according to equation (55). Then, taking into account the normalization condition (21) and values of $T^{(00)}_{[0,0]0}$ and $D^{(00)}_{[0]0,[0]0}(g)$,

$$T^{(00)}_{[0,0]0} = \frac{1}{\sqrt{3}}I \qquad D^{(00)}_{[0]0,[0]0}(g) = 1$$

we find that the density matrix has the form

$$\rho = \frac{1}{3}I + \sum_{Y,T,M} R^{(11)}_{YTM} T^{(11)}_{YTM}.$$

The first term in the above equation corresponds to the singlet term and the sum represents a contribution of the octet term. Let us suppose, for example, that the system is prepared in the state $|1\rangle$. The resulting density matrix is of the form

$$\rho = \tfrac{1}{3}I + \hat{Y} \tag{56}$$

i.e. in this case all coefficients $R^{(11)}_{YTM}$ vanish except for

$$R^{(11)}_{000} = \sqrt{\tfrac{2}{3}}.$$

In order to illustrate how the reconstruction formula (24) does work, we calculate the coefficient $R^{(11)}_{000}$ using this formula. It is necessary to take into account the following points:

(i) Since the vector $|1\rangle$ possesses the following quantum numbers (with respect to transformations from subgroups of $U$, $V$ and $T$ spins),

$$U = \tfrac{1}{2} \qquad U_3 = \tfrac{1}{2} \qquad V = \tfrac{1}{2} \qquad V_3 = \tfrac{1}{2} \qquad T = 0 \qquad T_3 = 0$$

we obtain in the notation

$$|1\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix} \qquad |2\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad |3\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

that

$$D(g)|1\rangle = \begin{bmatrix} \cos\frac{\theta_U}{2}\cos\frac{\theta_V}{2}\,\mathrm{e}^{\mathrm{i}(\phi_U+\phi_U+\psi_U)} \\ -\mathrm{i}\sin\frac{\theta_U}{2}\cos\frac{\theta_V}{2}\,\mathrm{e}^{\mathrm{i}(-\phi_U+\phi_U+\psi_U)} \\ -\mathrm{i}\sin\frac{\theta_V}{2}\,\mathrm{e}^{\mathrm{i}(-\phi_U+\psi_U)} \end{bmatrix} \tag{57}$$

where we have used equations (83)–(87) from appendix D. It follows from the above equation that the probability of detecting the state $|1\rangle$ in the transformed coordinate system is

$$\begin{aligned} w(g;\,(10)[-2/3,0]0) &= \cos^2\frac{\theta_U}{2}\cos^2\frac{\theta_V}{2} \\ w(g;\,(10)[1/3,1/2]1/2) &= \sin^2\frac{\theta_U}{2}\cos^2\frac{\theta_V}{2} \\ w(g;\,(10)[1/3,-1/2]1/2) &= \sin^2\frac{\theta_V}{2}. \end{aligned} \tag{58}$$

(ii) The $SU(3)\,D$-function we need is of the form (see appendix D)

$$D^{(11)}_{000,000}(\theta_U,\theta_V) = -\tfrac{1}{8} + \tfrac{3}{8}\cos\theta_U + \tfrac{3}{8}\cos\theta_V + \tfrac{3}{8}\cos\theta_U\cos\theta_V. \tag{59}$$

(iii) The necessary Clebsch–Gordan coefficient is

$$\left\langle (10) - \tfrac{2}{3}00,\,(11)000\middle|(10) - \tfrac{2}{3}00\right\rangle = \tfrac{1}{2}. \tag{60}$$

Substituting expressions (58)–(60) into equation (24) we obtain the result (56) as it should be.

### 6.2. Octet states

The self-conjugate, $A = A^*$, representation of the $SU(3)$ group (1, 1) is of $\dim A = 8$. The direct product of two representations (1, 1) contains the representation (1, 1) with multiplicity $k_B = 2$

$$(1,1) \otimes (1,1) = (0,0) \oplus (1,1) \oplus (1,1) \oplus (3,0) \oplus (0,3) \oplus (2,2). \tag{61}$$

Thus, to find coefficients $R^B_{[m_1]t_1}$ corresponding to representations (0, 0), (3, 0), (0, 3), (2, 2), we can use equation (24). Nevertheless, coefficients $R^{B,s}_{[m_1]t_1}$ correspond to a couple of degenerate representations (1, 1) appearing in equation (61), and we have to make use of the solution of the system (23) in the form (25). Let us consider, for example, a particular case when the reference state is the highest weight vector $|(1,1), [-1, 1/2], 1/2\rangle$. The index $s$ takes values $s = 1, 2$ while the index of inner multiplicity $t_2 = 0, 1$. One can show that the matrix elements $C_{T''s}$ of the matrix $C$ defined in equation (26), take the form [47–49]

$$C_{T''s} = \begin{bmatrix} & s=1 & s=2 \\ T''=0 & 1/2 & 1/\sqrt{20} \\ T''=1 & 1/\sqrt{12} & -\sqrt{3/20} \end{bmatrix}$$

such that $\det C = -1/\sqrt{15}$. Thus

$$R^{B,1}_{[m_1]t_1} = \frac{1}{2\pi^2} \int \mathrm{d}g \left[ \left( D^B_{[0]0,[m_1]t_1}(g) \right)^* + \frac{1}{\sqrt{3}} \left( D^B_{[0]1,[m_1]t_1}(g) \right)^* \right] w(g, (11)[-1, 1/2]1/2)$$

$$R^{B,2}_{[m_1]t_1} = \frac{\sqrt{5}}{6\pi^2} \int \mathrm{d}g \left[ \left( D^B_{[0]0,[m_1]t_1}(g) \right)^* - \sqrt{3} \left( D^B_{[0]1,[m_1]t_1}(g) \right)^* \right] w(g, (11)[-1, 1/2]1/2)$$

where $w(g, (11)[-1, 1/2]1/2)$ is the tomographic probability for a given reference state which corresponds to isospin $T = 1/2$.

## 7. Conclusions

We have developed a scheme of tomographic reconstruction of the Hermitian density matrix for quantum states which are associated with irreducible representations of unitary groups. Pure states of quantum systems, which are identified with basis vectors of irreducible representations of the corresponding group, are mapped on to tomographic probability distributions. We have applied this approach to two particular examples of the $SU(2)$ and $SU(3)$ groups. For the $SU(2)$ group, we have presented general different but equivalent expressions for the reconstruction of the density matrix in terms of the tomographic probability distribution. For the $SU(3)$ group, the reconstruction expressions have been obtained by taking into account the multiplicities in the Clebsch–Gordan decomposition of the tensor product of irreducible representations.

It should be pointed out that it is possible to interpret the suggested approach in two ways. One possibility is to consider the tomogram as an instrument (inversion formula) to reconstruct the quantum state identified with the Wigner function. In this case, there are additional problems. The employment of the inversion formula can create some errors. Whenever real experimental data are inserted, the result of reconstruction can exhibit artefacts of various origins. In quantum optics, a statistical approach to solve this problem was discussed in [50, 51]. Another possibility is to consider the tomogram as the final information on the quantum state. This means that one identifies the state with the tomogram. Using this method one can avoid the above problem.

The obtained expressions provide a possibility to interpret spinors as well as quarks, octets and other particle states as positive probability distributions, alternative to the association of these quantum states with vectors in Hilbert space. The transition probabilities between quantum spin states of the particles are expressed in terms of an integral containing the product of tomograms of corresponding states.

The approach developed can be generalized to more complicated constructions such as, for example, quantum groups.

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## Appendix A. $SU(3)$ group representations

In this appendix, we briefly describe the irreducible representations of the $SU(3)$ group and the corresponding matrix elements of finite transformations ($D$-functions) [47–49].

Generators of the $SU(3)$ group ($A_{ik}, i, k = 1, 2, 3$) satisfy the following commutation relations,

$$[A_{ik}, A_{pq}] = \delta_{kp}A_{iq} - \delta_{iq}A_{pk} \qquad A^{\dagger}_{ik} = A_{ki}$$

and the diagonal generators $A_{ii}$, which form an Abelian subalgebra $[A_{ii}, A_{pp}] = 0$, satisfy the following condition:

$$\sum_{p=1}^{3} A_{pp} = 0.$$

Thus, we can choose two independent diagonal generators in the following form:

$$H_\lambda = A_{11} - A_{22} \qquad H_\mu = A_{22} - A_{33}.$$

Each irreducible representation is labelled by its highest weight $\Lambda = (\lambda\mu)$, where $\lambda$ and $\mu$ are non-negative integers. Below we denote the signature of irreducible representation $A$ for a particular case of the $SU(3)$ group as the highest weight $\Lambda$. The corresponding (highest state) vector, which we denote as $|(\lambda\mu), H\rangle$, satisfies the following conditions:

$$A_{ik}|(\lambda\mu), H\rangle = 0 \qquad \text{if} \quad i < k$$
$$H_\lambda|(\lambda\mu), H\rangle = \lambda|(\lambda\mu), H\rangle$$
$$H_\mu|(\lambda\mu), H\rangle = \mu|(\lambda\mu), H\rangle.$$

The dimension of representation is given by

$$\dim(\lambda\mu) = \tfrac{1}{2}(\lambda + 1)(\mu + 1)(\lambda + \mu + 2).$$

## Appendix B. Description of basis

There are three canonical reductions of the $SU(3)$ group to different subgroups $U(1)\otimes SU(2)$. We consider the basis corresponding to each of these reductions.

### *B.1. T-basis*

This basis corresponds to the following reduction,

$$SU(3) \to U_Y(1) \otimes SU_T(2)$$

where

$$T_+ = A_{23} \qquad T_- = A_{32} \qquad T_0 = \frac{1}{2}H_\mu \tag{62}$$

are generators of the subgroup $SU_T(2)$, usually called the $T$-spin subgroup. Operators (62) satisfy standard $SU(2)$ commutation relations

$$[T_0, T_\pm] = \pm T_\pm \qquad [T_+, T_-] = 2T_0. \tag{63}$$

The only generator of the $U_Y(1)$ subgroup is

$$\hat{Y} = -\frac{1}{3}(2H_\lambda + H_\mu) \qquad [\hat{Y}, T_{\pm,0}] = 0$$

usually called the hypercharge operator.

In this case, we denote basis vectors of the irreducible representation $\Lambda$ by

$$|\Lambda, YTT_3\rangle \equiv |\Lambda, jTT_3\rangle \tag{64}$$

where quantum numbers $Y$, $T$, $T_3$ take the following values:

$$\begin{aligned} &Y = -\frac{1}{3}(2\lambda + \mu) + p + q \qquad && 0 \leqslant p \leqslant \lambda \qquad && 0 \leqslant q \leqslant \mu \\ &T = \frac{1}{2}(\mu + p - q) && -T \leqslant T_3 \leqslant T && j = \frac{1}{2}(p + q). \end{aligned} \tag{65}$$

### *B.2. U-basis*

This basis corresponds to the reduction

$$SU(3) \to U_Z(1) \otimes SU_U(2)$$

where generators of the $SU_U(2)$ subgroup ($U$-spin subgroup)

$$U_+ = A_{12} \qquad U_- = A_{21} \qquad U_0 = \frac{1}{2}H_\lambda$$

satisfy the commutation relations similar to (63) and the operator

$$Z = -\frac{1}{3}(H_\lambda + H_\mu) \qquad [Z, U_{\pm,0}] = 0$$

is the generator of the $U_Z(1)$ subgroup. We denote the corresponding basis vectors by

$$|\Lambda, ZUU_3\rangle \equiv |\Lambda, fUU_3\rangle \tag{66}$$

where

$$\begin{aligned} &Z = -\frac{1}{3}(\lambda + 2\mu) + k + l \qquad && 0 \leqslant k \leqslant \lambda \qquad && 0 \leqslant l \leqslant \mu \\ &U = \frac{1}{2}(\lambda + l - k) && -U \leqslant U_3 \leqslant U && f = \frac{1}{2}(k + l). \end{aligned} \tag{67}$$

### *B.3. V-basis*

This basis corresponds to the reduction

$$SU(3) \to U_X(1) \otimes SU_V(2)$$

where generators of the $SU_V(2)$ subgroup ($V$-spin subgroup)

$$U_+ = A_{13} \qquad U_- = A_{31} \qquad U_0 = \frac{1}{2}(H_\mu + H_\lambda)$$

satisfy the commutation relations similar to (63) and the operator

$$X = -\frac{1}{3}(H_\lambda - H_\mu) \qquad [X, V_{\pm,0}] = 0$$

is the generator of the $U_Z(1)$ subgroup. We denote the corresponding basis vectors by

$$|\Lambda, XVV_3\rangle \equiv |\Lambda, cVV_3\rangle \tag{68}$$

where

$$\begin{array}{lll} X = -\frac{1}{3}(\lambda-\mu)+e-g & 0 \leqslant e \leqslant \lambda & 0 \leqslant g \leqslant \mu \\ V = \frac{1}{2}(\mu-e-g) & -V \leqslant V_3 \leqslant V & c = \frac{1}{2}(\mu+e-g). \end{array} \tag{69}$$

## Appendix C. Transitions between T-, U- and V-bases

Each vector of canonical bases (64), (66) and (68) can be expressed in terms of any other basis vectors. The matrix elements of the corresponding transitions are as follows.

### *C.1. U–T transition*

One has the expression for the scalar product in terms of $3j$ symbols

$$\langle\Lambda, ZUU_3|\Lambda, YTT_3\rangle = \delta_{ff'}\delta_{U_3U_3'}(-1)^{\lambda+\mu}\sqrt{(2T+1)(2U+1)}\begin{Bmatrix} T & j & \mu/2 \\ U & j_1 & j_2 \end{Bmatrix} \tag{70}$$

where

$$\begin{array}{lll} j_1 = \frac{1}{2}(\lambda+\mu)-f' & j_2 = \frac{1}{2}\lambda+f'-j & f' = \frac{1}{2}\left(\frac{1}{2}\mu+j-T_3\right) \\ f = \frac{1}{2}(k+l) & j = \frac{1}{2}(p+q) & U_3' = \frac{1}{2}\left(\lambda+\frac{1}{2}\mu-3j-T_3\right) \end{array}$$

and $\left\{\begin{smallmatrix} T & j & \mu/2 \\ U & j_1 & j_2 \end{smallmatrix}\right\}$ denotes the $3j$ symbol.

### *C.2. V–T transition*

One has the expression for the scalar product

$$\langle\Lambda, XVV_3|\Lambda, YTT_3\rangle = \delta_{cc'}\delta_{V_3V_3'}(-1)^{\lambda-\mu/2+j-T}\sqrt{(2T+1)(2V+1)}\begin{Bmatrix} T & j & \mu/2 \\ V & j_3 & j_4 \end{Bmatrix} \tag{71}$$

where

$$\begin{array}{lll} j_3 = \frac{1}{2}(\lambda+\mu)-c' & j_4 = \frac{1}{2}\lambda+c'-j & c' = \frac{1}{2}\left(\frac{1}{2}\mu+j+T_3\right) \\ c = \frac{1}{2}(\mu+e-g) & V_3' = \frac{1}{2}\left(\lambda+\frac{1}{2}\mu-3j+T_3\right). & \end{array}$$

### *C.3. U–V transition*

One has the matrix element in the form

$$\langle\Lambda, ZUU_3|\Lambda, XVV_3\rangle = \delta_{ff'}\delta_{U_3U_3'}(-1)^{f+c+U+V}\sqrt{(2U+1)(2V+1)}\begin{Bmatrix} U & j_5 & \lambda/2 \\ V & c & j_6 \end{Bmatrix} \tag{72}$$

where

$$\begin{array}{lll} j_5 = \frac{1}{2}\left(\frac{1}{2}\lambda+\mu-c-V_3\right) & j_6 = \frac{1}{2}\lambda+\mu-c-j_5 & c' = \frac{1}{2}\left(\frac{1}{2}\mu+j+T_3\right) \\ U_3' = \frac{1}{2}\left(\frac{1}{2}\lambda+\mu-3c+V_3\right). & & \end{array}$$

## Appendix D. *D*-functions for the $SU(3)$ group

Let us consider an irreducible representation $\Lambda$ of the $SU(3)$ group ($g \to D(g)$, $g \in SU(3)$) with highest weight $\Lambda$. As a basis in the representation space $\mathcal{H}$ we choose $T$-basis (64). The action of $D(g)$ on the states (64) is defined according to

$$D(g)|\Lambda, YTT_3\rangle = \sum_{[m'],T'} D^{\Lambda}_{[m']T',[m]T}(g)|\Lambda, [m']T'\rangle$$

where $[m]$ is a collective index $[m] = (Y, T_3)$ and matrix elements

$$D^{\Lambda}_{[m]T,[m_1]T_1}(g) = \langle\Lambda, [m]T|D(g)|\Lambda, [m_1]T_1\rangle \tag{73}$$

define the $SU(3)$ $D$-function in the $T$-basis. To find an explicit expression for $D^{\Lambda}_{[m]T,[m_1]T_1}(g)$, we use the following parametrization for the $SU(3)$ group elements [52],

$$D^{\Lambda}(g) = D^U(\varphi_U, \theta_U)D^V(\varphi_V, \theta_V)D^T(\varphi_T, \theta_T)N(\psi_U, \psi_T) \tag{74}$$

where

$$D^U(\varphi_U, \theta_U) = \mathrm{e}^{\mathrm{i}\varphi_U U_0}\,\mathrm{e}^{\mathrm{i}\theta_U(U_+ + U_-)/2} \tag{75}$$

$$D^V(\varphi_V, \theta_V) = \mathrm{e}^{\mathrm{i}\varphi_V V_0}\,\mathrm{e}^{\mathrm{i}\theta_V(V_+ + V_-)/2} \tag{76}$$

$$D^T(\varphi_T, \theta_T) = \mathrm{e}^{\mathrm{i}\varphi_T T_0}\,\mathrm{e}^{\mathrm{i}\theta_V(T_+ + T_-)/2} \tag{77}$$

$$N(\psi_U, \psi_T) = \mathrm{e}^{\mathrm{i}\psi_U U_0}\,\mathrm{e}^{\mathrm{i}\psi_T T_0} \tag{78}$$

are the operators of finite rotations for $U$-, $V$- and $T$-spin subgroups, respectively. The Euler angles $\theta$, $\varphi$ and $\psi$ vary in the following domains:

$$\begin{array}{lll} 0 \leqslant \theta_U < \pi & 0 \leqslant \varphi_U < 2\pi & 0 \leqslant \psi_U < 4\pi \\ 0 \leqslant \theta_V < \pi & 0 \leqslant \varphi_V < 2\pi & \\ 0 \leqslant \theta_T < \pi & 0 \leqslant \varphi_T < 2\pi & 0 \leqslant \psi_T < 4\pi. \end{array}$$

Making use of equations (74)–(78) one can represent the matrix element (73) in the form

$$D^T_{Y'T'_3T',YT_3T}(g) = \sum_{Z,U_3,U}\sum_{X,V_3,V} D^U_{Y'T'_3T',ZU_3U}(\varphi_U, \theta_U)D^V_{ZU_3U,XV_3V}(\varphi_V, \theta_V) \times D^T_{XV_3V,YT_3T}(\varphi_T, \theta_T)N_{YT_3T}(\psi_U, \psi_T). \tag{79}$$

Diagonal matrix elements $N_{YT_3T}(\psi_U, \psi_T)$ have a simple form

$$N_{YT_3T}(\psi_U, \psi_T) = \mathrm{e}^{\mathrm{i}(U''_3\psi_U + T_3\psi_T)}$$

where

$$U''_3 = \tfrac{1}{2}\left(\mu - \tfrac{3}{2}Y - T_3\right).$$

The remaining matrix elements appearing in equation (79) can be expressed in terms of the usual matrix elements of the irreducible representations of the $SU(2)$ group. In fact, using connections between different bases (70)–(72), we find

$$D^U_{Y'T'_3T',ZU_3U}(\varphi_U, \theta_U) = \delta_{ff'}(-1)^{\lambda+\mu}\sqrt{(2T'+1)(2U+1)}\begin{Bmatrix} T' & j' & \mu/2 \\ U & j_1 & j_2 \end{Bmatrix}\mathrm{e}^{\mathrm{i}U'_3\varphi_U}d^U_{U'_3U_3}(\theta_U) \tag{80}$$

$$D^V_{ZU_3U,XV_3V}(\varphi_V, \theta_V) = \delta_{cj_3}(-1)^{c+f+V+U}\sqrt{(2V+1)(2U+1)}\begin{Bmatrix} U & f & \lambda/2 \\ V & j_3 & j_4 \end{Bmatrix}\mathrm{e}^{\mathrm{i}V'_3\varphi_V}d^V_{V'_3V_3}(\theta_V) \tag{81}$$

$$D^T_{XV_3V,YT_3T}(\varphi_T,\theta_T) = \delta_{jj_5}(-1)^{\lambda+\mu/2+j-T}\sqrt{(2V+1)(2T+1)}\begin{Bmatrix} T & j_5 & \mu/2 \\ V & j_6 & j_7 \end{Bmatrix}\mathrm{e}^{\mathrm{i}T_3''\varphi_T}d^T_{T_3''T_3}(\theta_T) \tag{82}$$

where

$$\begin{array}{ll}
j_1 = \frac{1}{2}(\lambda+\mu) - f' & j_2 = \frac{1}{2}\lambda + f' - j' \\
f' = \frac{1}{2}\left(\frac{1}{2}\mu + j' - T_3'\right) & U_3' = \lambda + \frac{1}{2}\mu - 3j' - T_3' \\
j_3 = \frac{1}{2}\left(\frac{1}{2}\lambda + \mu - f - U_3\right) & j_4 = \frac{1}{2}\lambda + \mu - f - j_3 \\
V_3' = \frac{1}{2}\left(\frac{1}{2}\lambda + \mu - 3f + U_3\right) & \\
j_5 = \frac{1}{2}\left(\frac{1}{2}\lambda + c - V_3\right) & j_6 = \frac{1}{2}(\lambda+\mu) - c \\
j_7 = \frac{1}{2}\lambda + c - j_5 & T_3'' = -\frac{1}{2}\left(\mu + \frac{1}{2}\lambda - 3c - V_3\right).
\end{array}$$

Relations between parameters $(Z, X, \Lambda)$ and $(f, c, j)$ are given by equations (65), (69) and (67).

For example, for the irreducible representation $\Lambda = (1, 0)$ we obtain from equations (80)–(82)

$$D^{(1,0)}(g) = D^U D^V D^T N \tag{83}$$

where

$$D^U = \begin{bmatrix} \cos\frac{\theta_U}{2}\,\mathrm{e}^{\mathrm{i}\phi_U/2} & -\mathrm{i}\sin\frac{\theta_U}{2}\,\mathrm{e}^{\mathrm{i}\phi_U/2} & 0 \\ -\mathrm{i}\sin\frac{\theta_U}{2}\,\mathrm{e}^{-\mathrm{i}\phi_U/2} & \cos\frac{\theta_U}{2}\,\mathrm{e}^{-\mathrm{i}\phi_U/2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{84}$$

$$D^V = \begin{bmatrix} \cos\frac{\theta_V}{2}\,\mathrm{e}^{\mathrm{i}\phi_V/2} & 0 & -\mathrm{i}\sin\frac{\theta_V}{2}\,\mathrm{e}^{\mathrm{i}\phi_V/2} \\ 0 & 1 & 0 \\ -\mathrm{i}\sin\frac{\theta_V}{2}\,\mathrm{e}^{-\mathrm{i}\phi_V/2} & 0 & \cos\frac{\theta_V}{2}\,\mathrm{e}^{-\mathrm{i}\phi_V/2} \end{bmatrix} \tag{85}$$

$$D^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\theta_T}{2}\,\mathrm{e}^{\mathrm{i}\phi_T/2} & -\mathrm{i}\sin\frac{\theta_T}{2}\,\mathrm{e}^{\mathrm{i}\phi_T/2} \\ 0 & -\mathrm{i}\sin\frac{\theta_T}{2}\,\mathrm{e}^{-\mathrm{i}\phi_T/2} & \cos\frac{\theta_T}{2}\,\mathrm{e}^{-\mathrm{i}\phi_T/2} \end{bmatrix} \tag{86}$$

$$N = \begin{bmatrix} \mathrm{e}^{\mathrm{i}\psi_U/2} & 0 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}(\psi_T-\psi_U)/2} & 0 \\ 0 & 0 & \mathrm{e}^{-\mathrm{i}\psi_T/2} \end{bmatrix}. \tag{87}$$

One can prove that matrix elements $D^{\Lambda}_{[m]T,[m_1]T_1}(g)$ satisfy the following normalization condition,

$$\int_{SU(3)} \mathrm{d}g\, D^{\Lambda}_{[m]T,[m_1]T_1}(g)\left(D^{\Lambda'}_{[m']T',[m_1']T_1'}(g)\right)^* = \frac{V_G}{\dim\Lambda}\delta_{\Lambda\Lambda'}\delta_{[m][m']}\delta_{[m_1][m_1']}\delta_{TT'}\delta_{T_1T_1'}$$

where $\mathrm{d}g$ is an invariant measure on the $SU(3)$ group

$$\mathrm{d}g = \sin\theta_U\cos^2\frac{\theta_V}{2}\sin\theta_V\sin\theta_T\,\mathrm{d}\theta_U\,\mathrm{d}\theta_V\,\mathrm{d}\theta_T\,\mathrm{d}\phi_U\,\mathrm{d}\phi_V\,\mathrm{d}\phi_T\,\mathrm{d}\psi_U\,\mathrm{d}\psi_T$$

such that

$$\int_{SU(3)}\mathrm{d}g = V_G$$

with $V_G = (1/2)(4\pi)^5$ being the group volume. Also, the matrix elements $D^{\Lambda}_{[m]T,[m_1]T_1}(g)$ satisfy the standard addition formula

$$\sum_{[m_1],T_1} D^{\Lambda}_{[m]T,[m_1]T_1}(g_1) D^{\Lambda}_{[m_1]T_1,[m_2]T_2}(g_2) = D^{\Lambda}_{[m]T,[m_2]T_2}(g_1 g_2).$$

In particular

$$\sum_{[m_1],T_1} D^{\Lambda}_{[m]T,[m_1]T_1}(g) \left(D^{\Lambda}_{[m_2]T_2,[m_1]T_1}(g)\right)^* = \delta_{[m][m_2]}\delta_{TT_2}.$$

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